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Replica-symmetry breaking for the Ashkin–Teller spin glass

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Abstract. The infinite-range Ashkin–Teller spin glass is studied by the replica method. Attention is focused on the implementation of Parisi's replica-symmetry-breaking scheme, and it is shown that, in general, two order-parameter functions are necessary to treat the problem. The change of behaviour of these functions is analysed in the interpolating region between two particular cases of the model: the four-state clock and Potts spin glasses. For most of this interpolating interval, the four-state clock behaviour dominates through the prevalence of one of the functions; only when the four-state Potts limit is approached, is it that both functions become of the same order of magnitude and then a crossover occurs.

1. Introduction

Generalizations of the infinite-range interaction Ising spin glass, i.e. of the Sherrington–Kirkpatrick (SK) model (Sherrington and Kirkpatrick 1975), have led to interesting results and open questions in the area of disordered magnetic systems (for reviews see Binder and Young 1986, van Hemmen and Morgenstern 1983, 1986, Mézard *et al.* 1987). In what concerns replica-symmetry breaking (Parisi 1979), the m-vector spin glasses (Gabay and Toulouse 1981) show a 'conventional' solution similar to that of the SK model (Gabay *et al.* 1982, Elderfield and Sherrington 1982). However, the same procedure when applied to quadrupolar glasses (Goldbart and Sherrington 1985) or to the p-state Potts model (Erzan and Lage 1983, Elderfield and Sherrington 1983a, b, c, Goldbart and Elderfield 1985), led to intriguing results. For the Potts case, the slope of the conventional Parisi function becomes negative for $p>2.82$, whereas the order-function breaking point exceeds unity for $p>4$. Such anomalies, which are prohibited by Parisi's theory (Parisi 1983, Houghton *et al.* 1983), appear as direct consequences of the absence of reflection symmetry in the spin variable. A step function, that is, the first stage in Parisi's replica-symmetry-breaking scheme, was proposed as the stable solution for these cases (Gross *et al.* 1985).

In order to test if such anomalies are present in other models with absence of reflection symmetry on the spin variable, Nobre and Sherrington (1986) studied the p-state clock model. For this model, the reflection symmetry is present (absent) for every p even (odd). The 'unconventional' solution (step function) appeared only for $p=3$, whereas for all other values of p (including the odd ones), the conventional Parisi function was found.

In the present work we study a spin-glass version of the Ashkin–Teller model (Ashkin and Teller 1943). This model contains as particular limits both $p=4$ clock and Potts spin glasses. With that, we wish to investigate how the Parisi solution goes from its conventional (four-state clock) to the step-function form (four-state Potts). We show that, in general, two order-functions are necessary to treat the problem. Throughout most of the interval between these two limiting cases, one of the functions prevails. Only when one approaches the four-state Potts limit is it that both functions become of the same order of magnitude; then the crossover between the conventional to the step-function form occurs.

In section 2 we present the spin-glass version of the Ashkin–Teller model we shall work with. A replica-symmetric solution and its different phases are described. In sections 3, 4 and 5, we apply replica-symmetry-breaking schemes and analyse the order functions, giving emphasis to the interpolating region between the four-state clock and Potts limits. Conclusions are drawn in section 6.

2. The Ashkin–Teller spin glass

The Ashkin–Teller spin glass is defined by the Hamiltonian

$$H=\sum_{(ij)}[J_{ij}(\rho_i\rho_j+\sigma_i\sigma_j)+L_{ij}\rho_i\rho_j\sigma_i\sigma_j] \tag{2.1}$$

where ρ_i, $\sigma_i(=\pm1)$ are Ising variables. In principle, the bond realizations $\{J_{ij}\}$ and $\{L_{ij}\}$ can be completely independent (Christiano and Goulart Rosa 1986, Moreira and Christiano 1992), but in the present paper we shall concentrate our analysis on the case in which

$$\{L_{ij}\}=\lambda\{J_{ij}\} \tag{2.2}$$

or in other words

$$H=\sum_{(ij)}J_{ij}(\rho_i\rho_j+\sigma_i\sigma_j+\lambda\rho_i\rho_j\sigma_i\sigma_j). \tag{2.3}$$

We shall work in the infinite-ranged spin-glass version (Sherrington and Kirkpatrick 1975), for which the summation is over all pairs (ij) and the J_{ij} are quenched random couplings distributed according to the probability

$$P(J_{ij})=(N/2\pi J^2)^{1/2}\exp[-N(J_{ij}-J_0/N)^2/2J^2]. \tag{2.4}$$

The Hamiltonian in (2.3) is a very rich one, containing as particular limits two well-known models as described below.

(a) $\lambda=0$: one recovers the four-state clock spin glass, that is, two independent Ising models, already discussed elsewhere (Nobre and Sherrington 1986, Nobre *et al.* 1989).

(b) $\lambda=1$: the Hamiltonian in (2.3) may be written as

$$H=-\sum_{(ij)}J_{ij}(4\delta_{\rho_i,\rho_j}\delta_{\sigma_i,\sigma_j}-1) \tag{2.5}$$

which gives in this limit, the four-state Potts model.

As far as replica-symmetry breaking is concerned, the limits $\lambda=0$ and $\lambda=1$ are very distinct from one another. The former presents a 'conventional' Parisi function, similar to that of the SK model, i.e. a monotonically increasing function followed by a plateau (Nobre and Sherrington 1986), whereas the later provides the peculiar step-function solution characteristic of Potts spin glasses (Gross *et al.* 1985). The main purpose of this paper is to analyse the intermediate region, $0\leqslant\lambda\leqslant1$, interpolating between these two limits.

Applying the usual replica trick (Edwards and Anderson 1975), one gets the free energy per spin in the thermodynamic limit ($N\rightarrow\infty$), as the extremal problem

$$\beta f=\lim_{n\rightarrow 0}\frac{1}{n}\min[g(q^{\alpha\beta}, v^{\alpha\beta})]. \tag{2.6}$$

The functional $g(q^{\alpha\beta}, v^{\alpha\beta})$ is given by

$$g(q^{\alpha\beta}, v^{\alpha\beta})=-\frac{n}{4}(\beta J)^2(2+\lambda^2)+(\beta J)^2\sum_{(\alpha\beta)}(q^{\alpha\beta})^2+\lambda^2\frac{(\beta J)^2}{2}\sum_{(\alpha\beta)}(v^{\alpha\beta})^2$$
$$-\ln \operatorname{Tr}\exp\{H_{\text{eff}}\} \tag{2.7a}$$

where

$$H_{\text{eff}}=(\beta J)^2\sum_{(\alpha\beta)}q^{\alpha\beta}(\rho^\alpha\rho^\beta+\sigma^\alpha\sigma^\beta)+\lambda^2(\beta J)^2\sum_{(\alpha\beta)}v^{\alpha\beta}s^\alpha s^\beta \tag{2.7b}$$

and $s^\alpha=\rho^\alpha\sigma^\alpha$. As usual, α and β are replica labels; $\alpha, \beta=1, \ldots, n$. The spins and trace are single-site and $\Sigma_{(\alpha\beta)}$ denotes a sum over pairs of different replicas $\alpha\neq\beta$. The two spin-glass parameters, $q^{\alpha\beta}$ and $v^{\alpha\beta}$, are given respectively by

$$q^{\alpha\beta}=\langle\rho^\alpha\rho^\beta\rangle=\langle\sigma^\alpha\sigma^\beta\rangle \tag{2.8a}$$

$$v^{\alpha\beta}=\langle s^\alpha s^\beta\rangle=\langle\rho^\alpha\sigma^\alpha\rho^\beta\sigma^\beta\rangle \tag{2.8b}$$

where the $\langle\rangle$ bracket denotes thermal averaging with respect to H_{eff}. Similarly to what happens for the Potts case, in obtaining the functional $g(q^{\alpha\beta}, v^{\alpha\beta})$ above, a convenient non-zero choice of J_0 was taken in order to ensure a stable spin-glass state at low temperatures (Elderfield and Sherrington 1983a, b, c, Goldbart and Elderfield 1985, Gross *et al.* 1985).

It is important to note that for the case $\lambda=1$, i.e. the four-state Potts limit, the free-energy functional in (2.7) is symmetric with respect to the variables ρ^α, σ^α and s^α. That gives

$$q^{\alpha\beta}=v^{\alpha\beta} \qquad (\lambda=1). \tag{2.9}$$

For $\lambda=0$, one has the four-state clock limit, for which the averages over the ρs decouple from those over σs due to the independence of these variables. This leads to a 'collinear' spin-glass state on the average, although small fluctuations from collinearity can be verified as a consequence of replica-symmetry breaking (Nobre *et al.* 1989). One has

$$v^{\alpha\beta}\cong(q^{\alpha\beta})^2 \qquad (\lambda=0) \tag{2.10}$$

where the equality should hold only in the replica symmetry approximation.

Although one knows that the replica symmetry approximation (Sherrington and Kirkpatrick 1975) leads to instabilities in the spin-glass phase (de Almeida and

Thouless 1978), it is instructive to start with such an ansatz. One may predict the correct boundaries of the paramagnetic phase and obtain, most of the time, a good approximation of the true phase diagram. In order to do this, we take

$$q^{\alpha\beta}=q \qquad v^{\alpha\beta}=v \qquad \text{(all pairs } (\alpha\beta)). \tag{2.11}$$

In this space, the free energy in (2.6) becomes

$$\beta f=-\tfrac{1}{4}(\beta J)^2[2(1-q)^2+\lambda^2(1-v)^2]-\int \mathrm{D}[x]\ln I_0 \tag{2.12}$$

where

$$q=\int \mathrm{D}[x]\left(\frac{I_1}{I_0}\right)^2=\int \mathrm{D}[x]\left(\frac{I_2}{I_0}\right)^2 \tag{2.13a}$$

$$v=\int \mathrm{D}[x]\left(\frac{I_3}{I_0}\right)^2. \tag{2.13b}$$

In the equations above

$$\int \mathrm{D}[x]\ldots=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(2\pi)^{1/2}}\frac{\mathrm{d}y}{(2\pi)^{1/2}}\frac{\mathrm{d}z}{(2\pi)^{1/2}}\mathrm{e}^{-x^2/2}\,\mathrm{e}^{-y^2/2}\,\mathrm{e}^{-z^2/2}\ldots, \tag{2.14}$$

and

$$I_0=4\cosh\theta\cosh\xi\cosh\phi+4\sinh\theta\sinh\xi\sinh\phi \tag{2.15a}$$

$$I_1=4\sinh\theta\cosh\xi\cosh\phi+4\cosh\theta\sinh\xi\sinh\phi \tag{2.15b}$$

$$I_2=4\cosh\theta\sinh\xi\cosh\phi+4\sinh\theta\cosh\xi\sinh\phi \tag{2.15c}$$

$$I_3=4\sinh\theta\sinh\xi\cosh\phi+4\cosh\theta\cosh\xi\sinh\phi \tag{2.15d}$$

$$\theta=\beta Jq^{1/2}x \qquad \xi=\beta Jq^{1/2}y \qquad \phi=\lambda\beta Jv^{1/2}z. \tag{2.15e}$$

The phase diagram is shown in figure 1. Besides the paramagnetic phase (P), one has two spin-glass phases, SG1 and SG2. SG1 ($q=0$; $v\neq 0$) is an Ising spin-glass phase in the variable $s=\rho\sigma$. The Ashkin–Teller spin-glass phase (SG2), where both q and v are non-zero, is the one on which we shall focus our attention throughout the following sections. In this phase, the results $v=q^2$ for $\lambda=0$, and $q=v$ for $\lambda=1$, follow trivially from equations (2.13). For $\lambda<1$ ($\lambda>1$) one has $q>v>0$ ($v>q>0$); the crossover between these two regimes occurs at the four-state Potts. As seen in figure 1, the phase SG2 is enhanced for $\lambda>1$; this is physically expected since in Hamiltonian (2.3) one has two Ising spin glasses (in the variables ρ and σ, respectively), each subjected to a bond Gaussian probability distribution with a finite J. These two systems become strongly correlated as λ increases and so the Ashkin–Teller phase (SG2) dominates the Ising phase (SG1).

In the following sections, we apply Parisi's replica-symmetry-breaking scheme (Parisi 1979) to the SG1 and SG2 phases, giving particular emphasis to the region of the Ashkin–Teller phase for which $0 \leqslant \lambda \leqslant 1$, interpolating between the four-state clock and Potts limits.

3. Replica-symmetry breaking in the spin-glass phases

From now on, we shall restrict our analysis to the region of the spin-glass phases (see figure 1) near the boundaries of the paramagnetic phase (P), i.e. $\eta \sim 0$, $1 \leqslant \lambda \leqslant \infty$

$$\eta = (T_{g1} - T)/T_{g1} \qquad T_{g1} = \lambda J \text{ (phase SG1)} \tag{3.1}$$

or $\tau \sim 0$, $0 \leqslant \lambda \leqslant 1$

$$\tau = (T_{g2} - T)/T_{g2} \qquad T_{g2} = J \text{ (phase SG2).} \tag{3.2}$$

Therefore, the free energy in (2.6) may be expanded perturbatively in powers of $q^{\alpha\beta}$

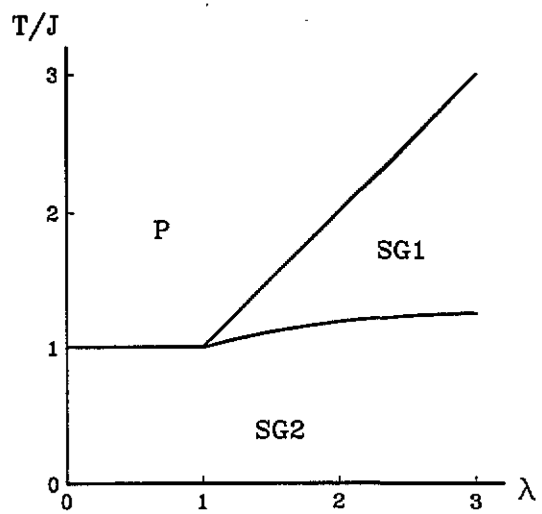

Figure 1. Phase diagram of the Ashkin–Teller spin glass in the replica symmetry approximation. The paramagnetic phase (P) ($q = v = 0$), has boundaries with two distinct spin-glass phases, namely SG1 and SG2. SG1 ($q = 0$; $v \neq 0$) is an Ising spin-glass phase in the variable $s = \rho\sigma$. SG2 ($q \neq 0$; $v \neq 0$) is the Ashkin–Teller spin-glass phase; for $\lambda < 1$ ($\lambda > 1$) one has $q > v > 0$ ($v > q > 0$), the crossover between these two regimes occurring at the four-state Potts limit ($\lambda = 1$), where $q = v > 0$.

and $v^{\alpha\beta}$; applying Parisi's replica-symmetry-breaking scheme (Parisi 1979), one gets the following free-energy functional

$$
\begin{aligned}
\beta f[q,v] = & -a_0 + a_2\langle q^2\rangle + b_2\langle v^2\rangle \\
& -a_3\left\{-\frac{1}{2}\lambda^2\langle q^2 v\rangle + \frac{1}{3}\int_0^1 \mathrm{d}x\left[xq^3(x) + 3q(x)\int_0^x \mathrm{d}y\, q^2(y)\right]\right. \\
& \left. +\frac{1}{6}\lambda^6\int_0^1 \mathrm{d}x\left[xv^3(x) + 3v(x)\int_0^x \mathrm{d}y\, v^2(y)\right]\right\} \\
& -a_4\left\{\frac{1}{3}\langle q^4\rangle - \frac{3}{4}\langle q^2\rangle^2 - \langle q\rangle^2\langle q\rangle^2 - \frac{1}{2}\int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y[q^2(x)-q^2(y)]^2\right. \\
& -\langle q\rangle\int_0^1 \mathrm{d}x\, q(x)\int_0^x \mathrm{d}y[q(x)-q(y)]^2 \\
& -\frac{1}{4}\int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y\int_0^x \mathrm{d}z[q(x)-q(y)]^2[q(x)-q(z)]^2 \\
& +\lambda^2\left(2\langle q\rangle\langle q^2 v\rangle + \int_0^1 \mathrm{d}x\, q(x)v(x)\int_0^x \mathrm{d}y[q(x)-q(y)]^2\right) \\
& +\frac{1}{2}\lambda^4\left(2\langle v\rangle\langle q^2 v\rangle + \int_0^1 \mathrm{d}x\, q^2(x)\int_0^x \mathrm{d}y[v(x)-v(y)]^2\right) \\
& +\frac{1}{2}\lambda^8\left(\frac{1}{3}\langle v^4\rangle - \frac{3}{4}\langle v^2\rangle^2 - \langle v\rangle^2\langle v^2\rangle - \frac{1}{2}\int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y[v^2(x)-v^2(y)]^2\right. \\
& -\langle v\rangle\int_0^1 \mathrm{d}x v(x)\int_0^x \mathrm{d}y[v(x)-v(y)]^2 \\
& \left.\left. -\frac{1}{4}\int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y\int_0^x \mathrm{d}z[v(x)-v(y)]^2[v(x)-v(z)]^2\right)\right\} + \ldots
\end{aligned}
\qquad (3.3)
$$

where

$$\langle q^k v^m\rangle = \int_0^1 \mathrm{d}x\, q^k(x)v^m(x). \qquad (3.4)$$

The coefficients above are given by

$$
\begin{aligned}
& a_0 = \ln 4 + \tfrac{1}{4}(\beta J)^2(2+\lambda^2) \qquad a_2 = \tfrac{1}{2}(\beta J)^2[(\beta J)^2 - 1] \\
& b_2 = \tfrac{1}{4}\lambda^2(\beta J)^2[\lambda^2(\beta J)^2 - 1] \qquad a_3 = (\beta J)^6 \qquad a_4 = (\beta J)^8.
\end{aligned}
\qquad (3.5)
$$

The phase transition P–SG1 ($\lambda\beta J = 1 + O(\eta)$) is properly described by setting $q(x)=0$; one reproduces the well-known results of the SK model, with the order-parameter function $v(x)$ presenting the conventional behaviour.

For the phase transition P–SG2 ($\beta J = 1 + O(\tau)$; $0 \leqslant \lambda \leqslant 1$), one has to consider the presence of both functions $q(x)$ and $v(x)$ by solving the extremal equations

$$\frac{\delta(\beta f)}{\delta q(x)} = 0 \qquad \frac{\delta(\beta f)}{\delta v(x)} = 0. \tag{3.6}$$

Taking derivatives of equations (3.6), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta(\beta f)}{\delta q(x)} = A[q, v]q'(x) + C[q, v]v'(x) = 0 \tag{3.7a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta(\beta f)}{\delta v(x)} = C[q, v]q'(x) + B[q, v]v'(x) = 0 \tag{3.7b}$$

where

$$\begin{aligned} A[q, v] = {} & 2a_2 + a_3\left\{\lambda^2 v(x) - 2xq(x) - 2\int_x^1 \mathrm{d}y\, q(y)\right\} + a_4\left\{2q^2(x) + \langle q^2\rangle\right. \\ & 4\langle q\rangle^2 + 6x\langle q\rangle q(x) + 3x^2q^2(x) - 6\langle q\rangle\int_0^x \mathrm{d}y\, q(y) - 6xq(x)\int_0^x \mathrm{d}y\, q(y) \\ & + x\int_0^x \mathrm{d}y\, q^2(y) + \int_x^1 \mathrm{d}y\, yq^2(y) + 2\int_0^x \mathrm{d}y\, q(y)\int_0^x \mathrm{d}z\, q(z) \\ & - 2\int_x^1 \mathrm{d}y\, q(y)\int_0^y \mathrm{d}z\, q(z) + \int_x^1 \mathrm{d}y\int_0^y \mathrm{d}z\, q^2(z) \\ & - \lambda^2\left(4\langle q\rangle v(x) + 6xq(x)v(x) - 4v(x)\int_0^x \mathrm{d}y\, q(y) + 2\int_x^1 \mathrm{d}y\, q(y)v(x)\right) \\ & \left. - \lambda^4\left(2\langle v\rangle v(x) + xv^2(x) - 2v(x)\int_0^x \mathrm{d}y\, v(y) + \int_0^x \mathrm{d}y\, v^2(y)\right)\right\} + \ldots \end{aligned} \tag{3.8a}$$

$$\begin{aligned} B[q, v] = {} & 2b_2 + \lambda^6 a_3\left\{-xv(x) - \int_x^1 \mathrm{d}y\, v(y)\right\} + a_4\left\{-\lambda^4\left(xq^2(x) + \int_x^1 \mathrm{d}y\, q^2(y)\right)\right. \\ & + \frac{1}{2}\lambda^8\left(2v^2(x) + \langle v^2\rangle + 4\langle v\rangle^2 + 6x\langle v\rangle v(x) + 3x^2v^2(x) - 6\langle v\rangle\int_0^x \mathrm{d}y\, v(y)\right. \\ & - 6xv(x)\int_0^x \mathrm{d}y\, v(y) + x\int_0^x \mathrm{d}y\, v^2(y) + \int_x^1 \mathrm{d}y\, yv^2(y) \\ & + 2\int_0^x \mathrm{d}y\, v(y)\int_0^x \mathrm{d}z v(z) \\ & \left.\left. - 2\int_x^1 \mathrm{d}y\, v(y)\int_0^y \mathrm{d}z\, v(z) + \int_x^1 \mathrm{d}y\int_0^y \mathrm{d}z\, v^2(z)\right)\right\} + \ldots \end{aligned} \tag{3.8b}$$

$$\begin{aligned} C[q, v] = {} & \lambda^2 a_3 q(x) - \lambda^2 a_4\left\{4\langle q\rangle q(x) + 3xq^2(x) - 4q(x)\int_0^x \mathrm{d}y\, q(y) + \int_0^x \mathrm{d}y\, q^2(y)\right\} \\ & - \lambda^4 a_4\left\{2\langle v\rangle q(x) + 2xq(x)v(x) - 2q(x)\int_0^x \mathrm{d}y\, v(y)\right\} + \ldots. \end{aligned} \tag{3.8c}$$

It is important to note that whenever $q'(x)\neq 0$ ($q'(x)=0$), equations (3.7) necessarily give $v'(x)\neq 0$ ($v'(x)=0$). For the x-region where both $q'(x)$ and $v'(x)$ are non-zero, one obtains from equations (3.7)

$$A[q,v]B[q,v]-C^2[q,v]=0. \tag{3.9}$$

Due to coupled terms, we were unable to solve equations (3.7) for $q(x)$ and $v(x)$ in general. In what follows, we treat the problem at first, in terms of step-function solutions; they represent the starting point towards Parisi's replica-symmetry-breaking scheme, providing a good qualitative behaviour for the true solutions. Afterwards, we discuss the general form of the Parisi's functions.

4. The step-function solutions

Throughout this section, we shall work with the step-function solutions as proposed by Gross *et al.* (1985) for the Potts spin glass. Such solutions can, in principle, be introduced for any value of λ (not only for $\lambda=1$), since they represent the first step in Parisi's replica-symmetry-breaking process.

As argued before, whenever $q'(x)\neq 0$ $(q'(x)=0)$, equations (3.7) necessarily imply $v'(x)\neq 0$ $(v'(x)=0)$. This gives the same breaking point for the two functions. Then one has

$$q(x)=q_{\rm m}\theta(x-\bar{x}) \qquad v(x)=v_{\rm m}\theta(x-\bar{x}) \tag{4.1}$$

where $\theta(y)$ is the usual step function, that is, 1 (0) for $y>0$ ($y<0$). Substituting (4.1) into (3.3) one gets a free energy per spin, $f(q_{\rm m}, v_{\rm m}, \bar{x})$, which after extremization, leads to a breaking point

$$\bar{x}=\bar{x}_0+\bar{x}_1+\ldots \tag{4.2a}$$

with

$$\bar{x}_0=\frac{3\lambda^2 q_{\rm m}^2 v_{\rm m}}{2q_{\rm m}^3+\lambda^6 v_{\rm m}^3} \tag{4.2b}$$

$$\bar{x}_1=6\frac{a_4}{a_3}\frac{(q_{\rm m}^4+\frac{1}{2}\lambda^8 v_{\rm m}^4)(\frac{1}{3}+\bar{x}_0-\frac{1}{2}\bar{x}_0^2)-2\lambda^2 q_{\rm m}^3 v_{\rm m}-\lambda^4 q_{\rm m}^2 v_{\rm m}^2}{2q_{\rm m}^3+\lambda^6 v_{\rm m}^3} \tag{4.2c}$$

In the Ising spin-glass phase (SG1) one has

$$q_{\rm m}=0 \qquad v_{\rm m}=\frac{2b_2}{\lambda^6 a_3}+O(\eta^2)=\eta+O(\eta^2) \tag{4.3a}$$

$$\bar{x}_0=0. \qquad \bar{x}=\bar{x}_1=\lambda^2\frac{a_4}{a_3}v_{\rm m}+O(\eta^2)=\eta+O(\eta^2) \tag{4.3b}$$

giving a small value for the breaking point $\bar{x}$, i.e. $\bar{x}\to 0$ as $\eta\to 0$, which is usually a characteristic of the conventional type of solution. Indeed, it is well known that for the Ising spin glass, the above solution is unstable and the correct one is a monotonically increasing function followed by a plateau (Parisi 1979).

In the Ashkin–Teller spin-glass phase (SG2), $\bar{x}_0$ varies in the interval [0, 1] for

$0 \leqslant \lambda \leqslant 1$; one can readily see that as $\lambda \to 0$, $\bar{x}_0 \to 0$, and $\bar{x}$ is dominated by $\bar{x}_1$, resulting in $\bar{x} \approx O(q_m)$. However, as λ increases, $\bar{x}_0$ increases up to the limit $\bar{x}_0 = 1$ for $\lambda = 1$ (cf equation (2.9)), such that in the four-state Potts limit one gets $\bar{x} = 1 + O(q_m)$.

Two distinct types of solutions may be obtained for the phase SG2. First, there are solutions with $v_m \ll q_m$:

$$q_m = \frac{a_2}{a_3} + O(\tau^2) = \tau + O(\tau^2) \tag{4.4a}$$

$$v_m = \frac{a_3}{1-\lambda^2} q_m^2 + O(\tau^3) = \frac{1}{1-\lambda^2} \tau^2 + O(\tau^3) \tag{4.4b}$$

$$\bar{x} = \left(\frac{3}{2} \frac{\lambda^2}{1-\lambda^2} a_3 + \frac{a_4}{a_3} \right) q_m + O(\tau^2) \tag{4.4c}$$

which are valid for most of the λ-interval of concern, except near the four-state Potts case. In this limit $((1-\lambda^2) \approx O(\tau))$, one obtains

$$q_m = v_m = \left(\frac{12}{35} \frac{a_2}{a_4} \right)^{1/2} + O(\tau) \tag{4.5a}$$

$$\bar{x} = 1 - \frac{7}{2} \frac{a_4}{a_3} q_m + O(\tau) \tag{4.5b}$$

or in other words, $q_m \approx O(\tau^{1/2})$ and $\bar{x} = 1 + O(\tau^{1/2})$, near $\lambda = 1$.

By investigating solutions for λ slightly greater than 1 we found a discontinuous behaviour of the one-step Parisi function between the spin-glass phases SG1 and SG2, insofar as in the former the step is at a low value of $\bar{x} (\bar{x} \ll 1)$ (see equation (4.3*b*)), whereas in the latter it is close to 1 $(1-\bar{x} \ll 1)$ (see equation (4.5*b*)) for λ near 1. This may be related to the fact that the p-state Potts spin glass when treated in replica symmetry approximation presents a first-order phase transition only for $p > 6$ (Elderfield and Sherrington 1983a), whereas the step-function solution is sufficient to lower the value of p for which that happens to $p > 4$ (Gross *et al.* 1985).

Numerical solutions were obtained for $0 \leqslant \lambda \leqslant 1$ and temperatures near T_{g2} (small τ), confirming the results described above. In figures 2 and 3 we show respectively, the increase of v_m with respect to q_m starting from $v_m \sim q_m^2$ $(\lambda = 0)$ up to $v_m = q_m$ $(\lambda = 1)$ and the behaviour of $\bar{x}$ with λ, both for two different values of τ. One can see that the ratio v_m / q_m together with the breaking-point $\bar{x}$ remain very small $(O(\tau))$ throughout most of the λ-interval; to lowest order, this region can be properly described in terms of the function $q(x)$ alone. Only as one approaches the four-state Potts regime is it that the second function $v(x)$ plays an important role; indeed, v_m increases very rapidly to $v_m \approx q_m$ and this effect is strongly correlated to a rapid growth of $\bar{x}$ towards 1.

In the appendix we perform a stability analysis of the solutions in (4.1); unfortunately, they lead to stability only in phase SG2 near the four-state Potts regime $((1-\lambda^2) \approx O(\tau))$; everywhere else near the boundaries of the paramagnetic phase (P) $(\eta \approx 0$, or $\tau \approx 0$ and $0 \leqslant \lambda < \bar{\lambda}(\tau)$; $\bar{\lambda}(\tau) = 1 - \tau/2 + O(q_m^3))$, they are unstable.

In spite of their simplicity, the solutions proposed in (4.1) are able to present some of the features of the true Parisi functions $q(x)$ and $v(x)$ (which come from the full replica-symmetry-breaking scheme) as we shall see in the following section.

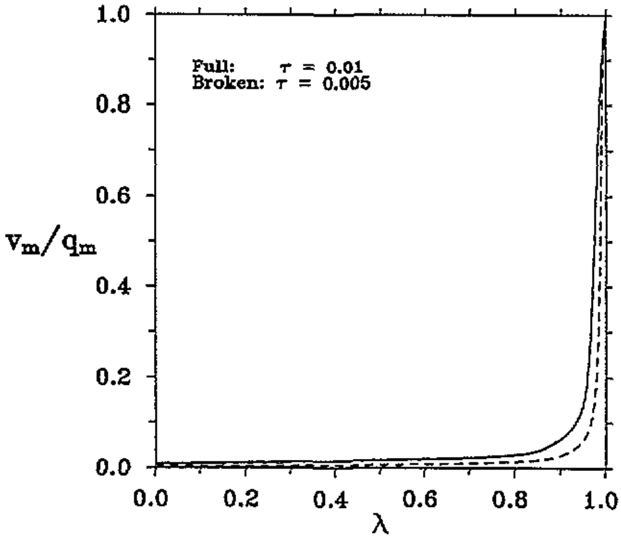

Figure 2. Plot of the ratio v_m/q_m as a function of $\lambda(0 \leq \lambda \leq 1)$ as obtained by solving equations (4.3) for two different values of τ.

5. The conventional Parisi solutions

Throughout this section we shall search for solutions of equations (3.6) which are *continuous*, *non-decreasing* functions, as proposed initially for the SK model (Parisi 1979), and generalized for the m-vector spin glasses (Gabay *et al.* 1982, Elderfield and Sherrington 1982). We call these the 'conventional' Parisi solutions, which, in the absence of a magnetic field, usually present a monotonically increasing part followed by a plateau.

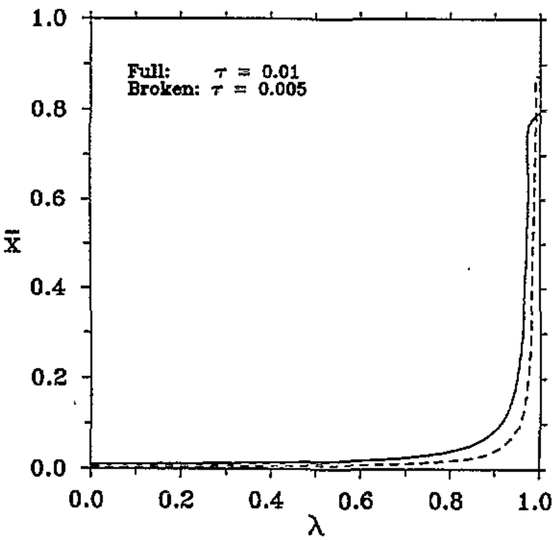

Figure 3. Plot of the breaking point $\bar{x}$ as a function of $\lambda(0 \leq \lambda \leq 1)$ as obtained by solving equations (4.3) for two different values of τ.

For the SG1 phase ($q(x)=0$), further derivatives of the second equation in (3.6) lead to a function $v(x)$ which recovers the well-known solution of the SK model. For the phase SG2, as long as the breaking point $\bar{x}$ remains small ($O(\tau)$), similarly to what happened in the previous section, one obtains solutions with $v_m \ll q_m$. By taking further derivatives of equations (3.7) one obtains

$$q(x)=\begin{cases} \frac{1}{2}x+O(\tau^2) & 0\leqslant x\leqslant \bar{x} \\ q_m & \bar{x}\leqslant x\leqslant 1 \end{cases} \tag{5.1a}$$

$$v(x)=\begin{cases} \frac{1}{4}\frac{1}{1-\lambda^2}x^2+O(\tau^3) & 0\leqslant x\leqslant \bar{x} \\ v_m & \bar{x}\leqslant x\leqslant 1 \end{cases} \tag{5.1b}$$

where q_m and v_m are, to lowest order, given, respectively, by equations (4.4*a*) and (4.4*b*). The region over which both $q'(x)$ and $v'(x)$ are non-zero is small in this case, i.e.

$$\bar{x}=2\tau+\lambda^2\frac{v'(t)}{q'(t)}+O(\tau^2)=2\tau+\frac{2\lambda^2\tau}{1-\lambda^2}+O(\tau^2). \tag{5.2}$$

The above solutions are valid for most of the relevant λ-interval, with the dominant behaviour dictated by $q(x)$ which is, to lowest order, λ-independent (see figure 4(a)). To this order, $q(x)$ is exactly the four-state clock Parisi function (Nobre and Sherrington 1986) and therefore, this whole λ-range is dominated by the four-state clock spin glass. In particular, for $\lambda=0$ one gets

$$v(x)=[q(x)]^2+O(\tau^3) \tag{5.3}$$

which is in agreement with prediction (2.10), ensuring a collinear spin-glass state at low temperatures (Nobre *et al.* 1989).

In the four-state Potts regime ($(1-\lambda^2\approx O(\tau)$), one gets an abrupt increase in $\bar{x}$ and the solution described above is not valid anymore. Indeed, equation (2.9) implies $q(x)\approx v(x)$, and then one has for the region where $q'(x)\neq 0$

$$q'(x)=\frac{a_3}{a_4(2-12x+3x^2)} \tag{5.4}$$

which gives an acceptable solution as long as x is smaller than $\bar{x}_c(\bar{x}_c\approx 0.174)$; we then, define $\lambda_c(\tau)$ as the value of λ for which $\bar{x}(\tau)=\bar{x}_c$. For $\lambda>\lambda_c(\tau)$, one obtains $q'(x)<0$, which according to the physical interpretation of the Parisi solution (Houghton *et al.* 1983, Parisi 1983), leads to a negative probability, becoming therefore unaccepable. As suggested by the simple analysis done in the previous section, near the four-state Potts spin glass the breaking point $\bar{x}(\tau)$ grows very rapidly from $O(\tau)$ to $O(1)$. Thus, *for values of λ in this regime ($(1-\lambda^2)\approx O(\tau)$) such that $\lambda<\lambda_c(\tau)$, the solutions are still* of the conventional type (see figure 4(b)), whereas for $\lambda>\bar{\lambda}(\tau)=1-\tau/2+O(q_m^3)$, the correct solutions are given by the step functions as described by equations (4.7) (see Figure 4(c)); their stability in this region is confirmed in the appendix.

Although we could not prove the coalescence of the values of λ for which the stability of the step functions breaks down, $\bar{\lambda}(\tau)$, and the slope of the conventional solutions diverge, $\lambda_c(\tau)$, we believe that to be true, as in the case of the Potts spin glass (Gross *et al.* 1985). Therefore, at the point $\lambda_c(\tau)$ at which $q'(x)\to\infty$, stable step-functions solutions should develop.

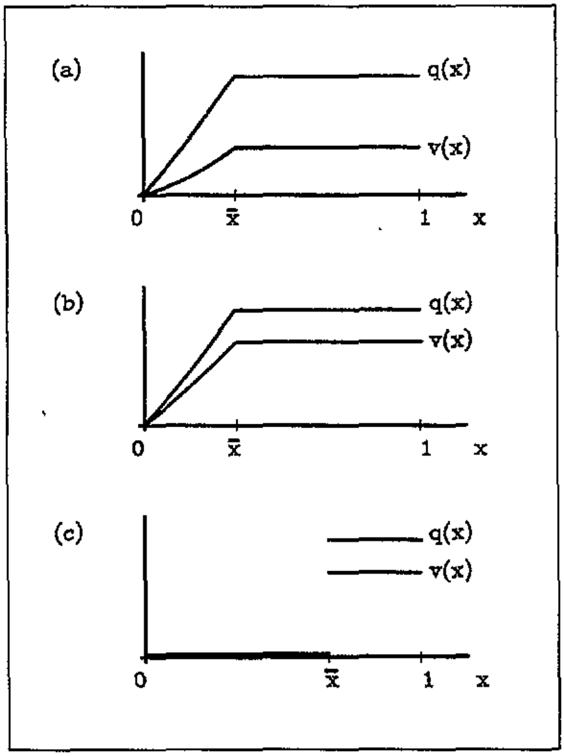

Figure 4. Parisi functions for the Ashkin–Teller spin glass: (a) picture for most of the λ-interval ($0 \leqslant \lambda \leqslant 1$); the function $q(x)$ prevails and therefore, the 4-state clock behaviour dominates; (b) as one approaches $\lambda = 1$, the two functions become of the same order of magnitude. The conventional solutions are still valid as long as λ is smaller than $\lambda_c(\tau)$; (c) this shows the four-state Potts regime where the step-function solutions are stable ($\bar{\lambda}(\tau) \leqslant \lambda \leqslant 1$; $\bar{\lambda}(\tau) = 1 - (\tau/2) + O(q_m^3)$).

6. Conclusion

We have studied an infinite-range spin-glass version of the Ashkin–Teller model for which the bond realizations $\{L_{ij}\}$ of the coupled term is related to those of the uncoupled terms ($\{J_{ij}\}$), by a factor λ. With this, one can obtain as particular limits the Ising as well as both $p = 4$ clock ($\lambda = 0$) and Potts ($\lambda = 1$) spin glasses. The phase diagram of the model was obtained within a replica symmetry approximation, presenting two distinct spin-glass phases, namely an Ising and an Ashkin–Teller one. We have focused our attention on the investigation of the changing of the Parisi solution from its conventional (four-state clock) to the step-function form (four-state Potts) in the Ashkin–Teller spin-glass phase. We have shown that in general, two order–functions are necessary to treat the problem. For most of the interpolating region between these two models, the four-state clock behaviour dominates by means of the prevalence of one of the functions and, therefore, conventional Parisi solutions are obtained. Only when one approaches the four-state Potts limit is it that both functions become of the same order of magnitude and then the crossover between the conventional to the step-function form occurs.

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Appendix 1. Stability analysis of the step-function solutions

In this appendix we look at the stability of the step-function solutions for the Ashkin–Teller spin glass as proposed in section 4. We start by evaluating the stability matrix

$$S[q,v]=\begin{bmatrix} \dfrac{\delta^2(\beta f)}{\delta q(y)\delta q(x)} & \dfrac{\delta^2(\beta f)}{\delta q(y)\delta v(x)} \\ \dfrac{\delta^2(\beta f)}{\delta v(y)\delta q(x)} & \dfrac{\delta^2(\beta f)}{\delta v(y)\delta v(x)} \end{bmatrix} \tag{A.1}$$

which can be calculated from (3.3) to give

$$\begin{aligned} S[q,v]=&\begin{bmatrix} -2a_3q(y) & -2\lambda^2a_4q^2(y)-2\lambda^4a_4q(x)v(y) \\ -2\lambda^2a_4q^2(y)-2\lambda^4a_4q(x)v(y) & -\lambda^6a_3v(y) \end{bmatrix}\theta(x-y) \\ &+\begin{bmatrix} -2a_3q(x) & -2\lambda^2a_4q(x)q(y)-2\lambda^4a_4q(x)v(x) \\ -2\lambda^2a_4q(x)q(y)-2\lambda^4a_4q(x)v(x) & -\lambda^6a_3v(x) \end{bmatrix}\theta(y-x) \\ &+\begin{bmatrix} A[q,v] & C[q,v] \\ C[q,v] & B[q,v] \end{bmatrix}\delta(x-y)+O(q_m^3) \end{aligned} \tag{A.2}$$

where $A[q,v]$, $B[q,v]$ and $C[q,v]$ are expressed in equations (3.8). To ensure stability, $S[q,v]$ must be negative definite, or in other words, all its eigenvalues should be negative (Thouless *et al.* 1980).

For the solutions in (4.1) our stability matrix takes the form

$$\begin{aligned} S(q_m,v_m,\bar{x},x,y)=&\begin{bmatrix} -2a_3q_m & -2\lambda^2a_4q_m(q_m+\lambda^2v_m) \\ -2\lambda^2a_4q_m(q_m+\lambda^2v_m) & -\lambda^6a_3v_m \end{bmatrix}\theta(x-\bar{x})\theta(y-\bar{x}) \\ &+\begin{bmatrix} A_+\theta(x-\bar{x})+A_-\theta(\bar{x}-x) & C_+\theta(x-\bar{x})+C_-\theta(\bar{x}-x) \\ C_+\theta(x-\bar{x})+C_-\theta(\bar{x}-x) & B_+\theta(x-\bar{x})+B_-\theta(\bar{x}-x) \end{bmatrix}\delta(x-y) \\ &+O(q_m^3) \end{aligned} \tag{A.3}$$

where $A_\pm$, $B_\pm$ and $C_\pm$ can be obtained by evaluating equations (3.8) for the step-function solutions in (4.1) at either side of the discontinuity ($x=\bar{x}\pm\varepsilon$; $\varepsilon\to 0$). Neglecting terms $O(q_m^3)$ one obtains

$$\begin{aligned} &A_+(q_m,v_m,\bar{x})=2a_2+a_3(\lambda^2v_m-2q_m)+a_4[q_m^2(7-2\bar{x})-6\lambda^2q_mv_m-\lambda^4v_m^2(2-\bar{x})] \\ &B_+(q_m,v_m,\bar{x})=2b_2-\lambda^6a_3v_m+a_4[-\lambda^4q_m^2+\tfrac{1}{2}\lambda^8v_m^2(7-2\bar{x})] \\ &C_+(q_m,v_m,\bar{x})=\lambda^2a_3q_m-\lambda^2a_4q_m^2(4-\bar{x})-2\lambda^4a_4q_mv_m \\ &A_-(q_m,v_m,\bar{x})=2a_2-2a_3q_m(1-\bar{x})+a_4(1-\bar{x})[q_m^2(5-3\bar{x})-2\lambda^2q_mv_m] \\ &B_-(q_m,v_m,\bar{x})=2b_2-\lambda^6a_3v_m(1-\bar{x})+a_4(1-\bar{x})[-\lambda^4q_m^2+\tfrac{1}{2}\lambda^8v_m^2(5-3\bar{x})] \\ &C_-(q_m,v_m,\bar{x})=0. \end{aligned} \tag{A.4}$$

We restrict our analysis to the longitudinal modes, similarly to the treatment of Thouless *et al.* (1980) for the SK model. In this case, one has to solve the set of eigenvalue equations,

$$\sum_k \int_0^1 \mathrm{d}y\, S_{kj}\Phi_k(y) = \mu\Phi_j(x) \qquad (j=1,2) \tag{A.5}$$

where $\Phi_k(x)$ $(k=1,2)$ denote components of a two-dimensional vector

$$\Phi(x) \equiv \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix}. \tag{A.6}$$

One can readily see that the set of vectors

$$\begin{pmatrix} f_+(x) \\ g_+(x) \end{pmatrix} \qquad \begin{pmatrix} \kappa_+\theta(x-\bar{x}) \\ \xi_+\theta(x-\bar{x}) \end{pmatrix} \qquad \begin{pmatrix} f_-(x) \\ g_-(x) \end{pmatrix} \qquad \begin{pmatrix} \kappa_-\theta(\bar{x}-x) \\ \xi_-\theta(\bar{x}-x) \end{pmatrix} \tag{A.7}$$

where $\kappa_\pm$ and $\xi_\pm$ are arbitrary constants and $f_+(x)$, $g_+(x)$ ($f_-(x)$, $g_-(x)$) are vanishing functions for $x<\bar{x}$ ($x>\bar{x}$), restricted to

$$\int_0^1 \mathrm{d}x\, f_\pm(x) = \int_0^1 \mathrm{d}x\, g_\pm(x) = 0 \tag{A.8}$$

do form a complete set. They are orthogonal to each other and any two-dimensional vector may be expressed as linear combinations of them.

Substituting the eigenvectors (A.7) into (A.5), one obtains that the condition for stability is fulfilled only near the four-state Potts limit. This happens because a double-degenerate eigenvalue

$$\mu = A_-(q_{\mathrm{m}}, v_{\mathrm{m}}, \bar{x}) \tag{A.10}$$

corresponding, respectively, to the third and fourth eigenvectors in (A.7), becomes positive as one decreases λ away from $\lambda=1$, signalling the instability of solutions (4.1). Indeed, this eigenvector may be expressed in this region as

$$\mu = \frac{2}{5}\left(\frac{1-\lambda^2}{\lambda^2} - \tau\right) + O(q_{\mathrm{m}}^3) \tag{A.11}$$

which gives a critical value for λ

$$\bar{\lambda} = 1 - \frac{\tau}{2} + O(q_{\mathrm{m}}^2) \tag{A.12}$$

below which the step-function solutions are unstable.

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